**Chap 4: Fourier Series**

1. **Fourier Series Expansion**
2. **Fourier Series Expansion of functions of period T (general case)**

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| Periodic Function | * A periodic function has its image values repeated at regular intervals in its domain: * The period of the function is the interval between 2 successive replicas: | |
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| Fourier Theorem | * Fourier series is an expansion of a periodic function of period in which the base set is the set of sine functions, giving an expanded representation of the form:   + : is the nth harmonic   + is the amplitude of the nth harmonic   + : is the frequency of   + : is the phase angle; measuring the lag or lead of the nth harmonic with reference to a pure sine wave of the same frequency * **Fourier Series Expansion:**      * + are the Fourier coefficients, given by the Euler formulae:   + A periodic function may be specified in a piecewise fashion:   Then: | |

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| Example | Obtain the Fourier series expansion of the periodic function of period defined by: | |
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| * The Fourier coefficient : * The Fourier coefficient : * The Fourier coefficient : * Hence the Fourier Series Expansion of is: | |

1. **Fourier Series Expansion of function of period**

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| Functions of period | * If the periodic function has a period then & the series becomes:   + Where are the Fourier coefficients: | |
| Example | Obtain the Fourier series expansion of the periodic function of period defined by: | |
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| * We have: * The Fourier coefficients : * The Fourier coefficients : * The Fourier coefficients : * Hence the Fourier Series Expansion of is: | |

1. **Fourier Series Expansion of odd & even functions**

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| Even & Odd Functions |  | | | |  |
| * The properties of even and odd functions:   + The sum of 2 (or more) odd functions = odd function   + The product of 2 even functions = even function   + The product of 2 odd functions = even function   + The product of an odd & an even function = odd function   + The derivative of an even function = odd function   + The derivative of an odd function = even function | | | |
| * **Fourier Series Expansion of even & odd function:**   + The Fourier series for an **even** function is a pure cosine series; it contains no sine terms.  |  |  | | --- | --- | |  |  |  * + The Fourier series for an **odd** function is a pure sine series; it contains no cosine terms.  |  |  | | --- | --- | |  |  | | | | | |
| Example | Obtain the Fourier series expansion of the periodic function of period defined by: | | | | |
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| * is an odd function, therefore: * The Fourier coefficient: : * Thus the Fourier Series Expansion of is: | | | | |
| Obtain the Fourier series expansion of the periodic function of period defined by: | | | | |
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| * is an even function, therefore: * The Fourier coefficient: : * The Fourier coefficient: : * Thus the Fourier Series Expansion of is: | | | | |
| Obtain the Fourier series expansion of the periodic function of period defined by: | | | | |
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| * is an even function, therefore: * The Fourier coefficient: : * The Fourier coefficient: : * Thus the Fourier Series Expansion of is: | | | | |

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| Linearity Property | * If we have:   is a periodic functions of period  is a periodic functions of period  is a periodic functions of period   * Then the Fourier series expansion of is: |
| Example | Obtain the Fourier series expansion of the periodic function of period defined by:   * The Fourier Series Expression of : * The Fourier Series Expression of the odd function :   Where:   * Using the linearity property : |

1. **Convergent of Fourier Series**

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| Convergence of Fourier Series | * **Dirichlet’s conditions**: If is a bounded periodic function that in any period has   + A finite number of isolated maxima and minima, and   + A finite number of points of finite discontinuity   Then the Fourier series expansion of converges to at all points where is continuous and to the average of the right- and left-hand limits of at points where is discontinuous (that is, to the mean of the discontinuity). |
| Example |  |

1. **Functions defined over a Finite Interval**
2. **Full-range Series**

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| Full-range Series | * Suppose the function is defined only over a finite interval * To find the **Full-range Fourier series** (a series contains both cosine & sine terms) representation of , we define the **periodic extension**  of by |
| Example | Find a full-range Fourier series expansion of valid in the finite interval     * The periodic function is defined by:        * Since is a periodic function with period 4, it has a convergent Fourier series expansion. * The Fourier coefficient : * The Fourier coefficient : * The Fourier coefficient : * The Fourier series expansion of is: * Since for , it follows that this Fourier series is representative of within this interval, so that: |

1. **Half-range cosine and sine series**

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| Half-range Series | * Suppose the function is defined only over a finite interval * It’s **even periodic extension**  is the **even periodic function** defined by:   has a convergent Fourier series representation consisting of **cosine terms only**   |  |  | | --- | --- | |  |  |      * It’s **odd periodic extension**  is the **odd periodic function** defined by:   has a convergent Fourier series representation consisting of **sine terms only**   |  |  | | --- | --- | |  |  | |
| Example | **For the function defined only in the interval . Find:**  **(a) a half-range cosine series expansion**  **(b) a half-range sine series expansion.**   1. Half-range cosine series. Define the periodic function by  * Since is an even periodic function with period 8, it has a convergent Fourier series expansion: * The Fourier coefficient : * The Fourier coefficient : * The Fourier series expansion of is: * Since for , it follows that this Fourier series is representative of within this interval. Therefore, the half-range cosine series expansion of is:  1. Half-range sine series. Define the periodic function by  * Since is an odd periodic function with period 8, it has a convergent Fourier series expansion: * The Fourier coefficient : * The Fourier series expansion of is: * Since for , it follows that this Fourier series is representative of within this interval. Therefore, the half-range sine series expansion of is: |

1. **Differentiation and integration of Fourier series**
2. **Integration of a Fourier series**

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| Integration | If satisfies Dirichlet’s conditions in the interval and has a Fourier series expansion:  Then for the integration of is:  The presence on the right-hand side is clearly not a Fourier series expansion of the integral on the left-hand side. So we rearrange the result to be a Fourier series expansion of the function: |
| Example | Obtain the integrated Fourier series expansion of the periodic function of period defined by:   * From the previous example, the Fourier series expansion of is: * Integrating this result between the limits and * Rearranging this result: * The Fourier series expansion is: |
| Obtain the integrated Fourier series expansion of the periodic function of period defined by:   * From the previous example, the Fourier series expansion of is: * First, integrating this result between the limits and * Second, integrating this result between the limits and , we have: * The function: * Has a Fourier series expansion |

1. **Differentiation of a Fourier series**

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| Differentiation | If is continuous everywhere and satisfies Dirichlet’s conditions and has a Fourier series expansion  Then if satisfies Dirichlet’s conditions, the differentiation of is: |
| Example | Obtain the differentiated Fourier series expansion of the periodic function of period defined by:   * From the previous example, the Fourier series expansion of is: * Since is continuous within and at the end points of the interval . We have: |

1. **Complex form of Fourier series**
2. **Complex representation**

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| Complex representation | * The complex form of sine & cosine:  |  |  | | --- | --- | | Sine |  | | Cosine |  | | Exponential |  |  * The complex or exponential form of the Fourier series expansion of the function of period :  |  |  | | --- | --- | |  |  |  * If the value of then: * If the value of then: | |
| Example | Find the complex form of the Fourier series expansion of the periodic function defined by | |
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| * The complex Fourier coefficient : * Now we have: * So: * The complex form of the Fourier series expansion | |

1. **The multiplication theorem and Parseval’s theorem**

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| Multiplication theorem | If and are 2 periodic functions having the same period then:   |  |  | | --- | --- | |  | & = the coefficients in the complex Fourier series expansions of and | |  | & & & = the coefficients in the Fourier series expansions of and | |
| Parseval’s theorem | If is a periodic function with period T then:   |  |  | | --- | --- | |  | = the coefficients in the complex Fourier series expansions of | |  | & = the coefficients in the Fourier series expansions of |  * The root mean square (RMS): |
| Example | A periodic function , of period , is defined  Using Parseval’s theorem, prove that :   * The coefficient : * The coefficient : * The coefficient : * Using the Parseval’s theorem: |

1. **Orthogonal functions**
2. **Definition**

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| Definition | * A set of complex functions each of which is piecewise-continuous on is an **orthogonal** set on this interval if   Where   * + is the complex conjugate of   + is a constant * An orthogonal set is an **orthonormal** set if each of its components is also normalized, that is: |
| Example | Verify that the set of complex exponential functions  is an orthogonal set on the interval   * When : * When : * When : * We have: * Therefore, the set is an orthogonal set on the interval |

1. **Generalized Fourier series**

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| Definition | * If:   + - is a piecewise-continuous function on the interval and     - is an **orthogonal set** on this interval * Then the generalized Fourier series of with respect to the basis set is:   Where:   * + : is the generalized Fourier coefficients |